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# Self-similar evolution in nonlocal nonlinear media

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**The self-similar propagation of optical beams in a broad class of nonlocal, nonlinear optical media is studied utilizing a generic system of coupled equations with linear gain. This system describes, for instance, beam propagation in nematic liquid crystals and optical thermal media. It is found, both numerically and analytically, that the nonlocal response has a focusing effect on the beam, concentrating its power around its center during propagation. In particular, the beam narrows in width and grows in amplitude faster than in local media, with the resulting beam shape being parabolic. Finally, a general initial localised beam evolves to a common shape.** 

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Let us consider the propagation of an optical beam in a general class of media with a nonlinear, nonlocal response, examples being thermal media [1, 2], plasmas [3], atomic vapors [4] and nematic liquid crystals [5, 6]. The response is nonlocal in the sense that the medium response to the optical beam extends far beyond the waist of the beam. In the paraxial, slowly varying envelope approximation, a general system of equations describing this propagation is [1, 5, 6]

$$i\frac{\partial\psi}{\partial z}+\frac{d_0}{2}\frac{\partial^2\psi}{\partial x^2}+2\theta\psi=i\gamma\psi, \tag{1a}$$

$$\nu\frac{\partial^2\theta}{\partial x^2}-2q\theta=-2|\psi|^2. \tag{1b}$$

Here, $\psi=\psi(x,z)$ is the complex valued, slowly varying envelope of the optical electric field and $\theta=\theta(x,z)$ is the medium response, the optically induced deviation of the director angle for a nematic or the temperature for a thermal medium, for instance. The propagation direction is $z$. The nonlocality $\nu$ measures the strength of the response of the medium, with a highly nonlocal response corresponding to $\nu$ large, as assumed here. The parameter $q$ is related to the square of the applied static electric field which pre-tilts the nematic dielectric [6–8] or the effective thermal response length in the $z$ direction [1], for instance. The parameter $\gamma>0$ measures the strength of the gain in the medium. This gain can be due to the inclusion of suitable dyes in a nematic [9] or the inclusion of nano-particles in a thermal medium [10]. Finally, when $d_0>0$ the system exhibits a focusing response and has bright soliton solutions, while for $d_0<0$ it has a defocusing response and dark and anti-dark soliton solutions [11, 12] (see also [13–15] for higher dimensional settings). A nematic typically has a focusing response, but the addition of an azo-dye can shift the response to defocusing [11]. Hereafter, we study the defocusing case, $d_0<0$, and fix $d_0=-1$.

The generic Eqs. (1), while $(1+1)$ dimensional, have no known general solitary wave solution, just isolated solutions for fixed parameter values [16]. To date, solutions have been found using numerical or approximate methods [5]. In this work, self-similar solutions of Eqs. (1) will be found. Self-similar phenomena have been both theoretically predicted and experimentally observed in ultrafast nonlinear optics [17], optical fibers [18], waveguide amplifiers [19], mode-locked lasers [20], and many other areas [21]. The generic system used to describe nonlinear beam propagation in optical media is the nonlinear Schrödinger (NLS) equation and its variants. In the local limit, $\nu\to 0$, the system (1) reduces to the NLS equation. In this limit exact solutions have been found for various nonlinear media [19, 22] with distributed coefficients. A similarity transformation is used to eliminate the distance-dependent coefficients of the equations, resulting in a regular, constant coefficient NLS system whose exact soliton solution is used to determine the general self-similar solution of the original system. Note, however, that for this to work certain restrictions on the coefficients have to be assumed and the method does not apply to self-similar evolution for arbitrary (decaying) initial data.

Self-similar solitary waves have been asymptotically derived for the defocusing NLS equation with linear gain [23]. This beam shape represents a type of nonlinear attractor towards which a rather general shaped input beam tends to after sufficient propagation distance. Such self-similar beams are often termed "similaritons" [24]. However, these structures may not be observed for the focusing NLS equation due to the onset of modulation instability (MI), which leads them to disintegrate during propagation. Similaritons are fundamentally different from NLS soliton solutions (which are dark solitons in the defocusing regime) in that they decay at infinity and are highly chirped. However, they share a common characteristic: they are shape preserving and resistant to wave breaking. Furthermore, it has also been shown recently that similaritons can be manipulated to the extent of shape management [25, 26]. This work will derive similariton solutions of the defocusing equations

(1). These solutions illustrate the major effects that nonlocality has on self-similar evolution. It is expected that these are general long-term solutions of these equations, as is general for similarity solutions, in the presence of gain.

We start by evolving a unit Gaussian, $\psi(x,0) = \exp(-x^2)$ with $q = 1$, $\nu = 50$ and $\gamma = 0.05$, as shown in Fig. 1 (in what follows we take $q = 1$). Here, we utilize the method of Ref. [27] to integrate Eqs. (1) in $z$.

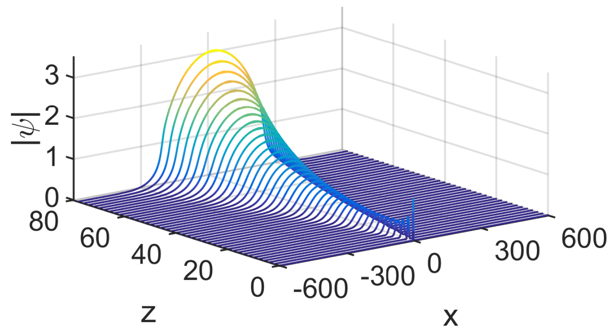


**Fig. 1.** (Color Online) Complete evolution of a unit Gaussian under Eqs. (1), with $q = 1$, $\nu = 50$ and $\gamma = 0.05$.

Clearly, in the early stage of the evolution, the initial unit Gaussian spreads and decreases in amplitude, but then the gain and nonlinear terms force the beam to undergo self-similar evolution with both its amplitude and width increasing, illustrating evolution to a self-similar solution. The key question is the rate at which this happens, the nature of the self-similar solution and whether any initial condition will evolve to it. We investigate these issues below.

First, we consider the effect of nonlocality and the profile of the initial beam. We take $\gamma = 0.05$ and find the solution of Eqs. (1) for different values of $\nu$ and two distinct initial beams, as shown in Fig. 2. As $\nu$ increases, the beam becomes "focused" around its center, making its amplitude higher and width narrower. However, while nonlocality induces a focusing effect, a (parabolic) shape preserving structure is formed, regardless of the initial shape (bottom panel of Fig. 2), the main difference being the distance at which this particular profile is formed. Also, it is apparent from Fig. 2 that, as the amplitude increases, the width of the beam shrinks. We show below that there is a direct relationship between the two resulting from the conservation of beam power.

There are two parameters we need to associate with beam evolution: gain and nonlocality. We first fix the nonlocality parameter $\nu = 50$ and vary the gain parameter $\gamma$, and then fix the gain at $\gamma = 0.05$ and vary $\nu$. The results are shown in Fig. 3 top and bottom, respectively.

In the top figure we see that the gain, as expected, enhances the rate at which the amplitude grows, but in all cases the nonlocal solution evolves faster than its local NLS counterpart ($\nu = 0$). In the bottom figure, it is clear that the higher the nonlocality, the higher the growth rate of the amplitude and, in fact, there is an abrupt change in the evolution from the local limit $\nu = 0$.

We now investigate this self-similar behaviour analytically. Unfortunately, as discussed above, Eqs. (1) have a very limited set of mathematical tools one can use to find exact solutions. In the past, and for similar problems for NLS-type equations, a Lagrangian formulation assuming Gaussian test functions has proved useful [5, 16, 28, 29]. Notably, this approach has also

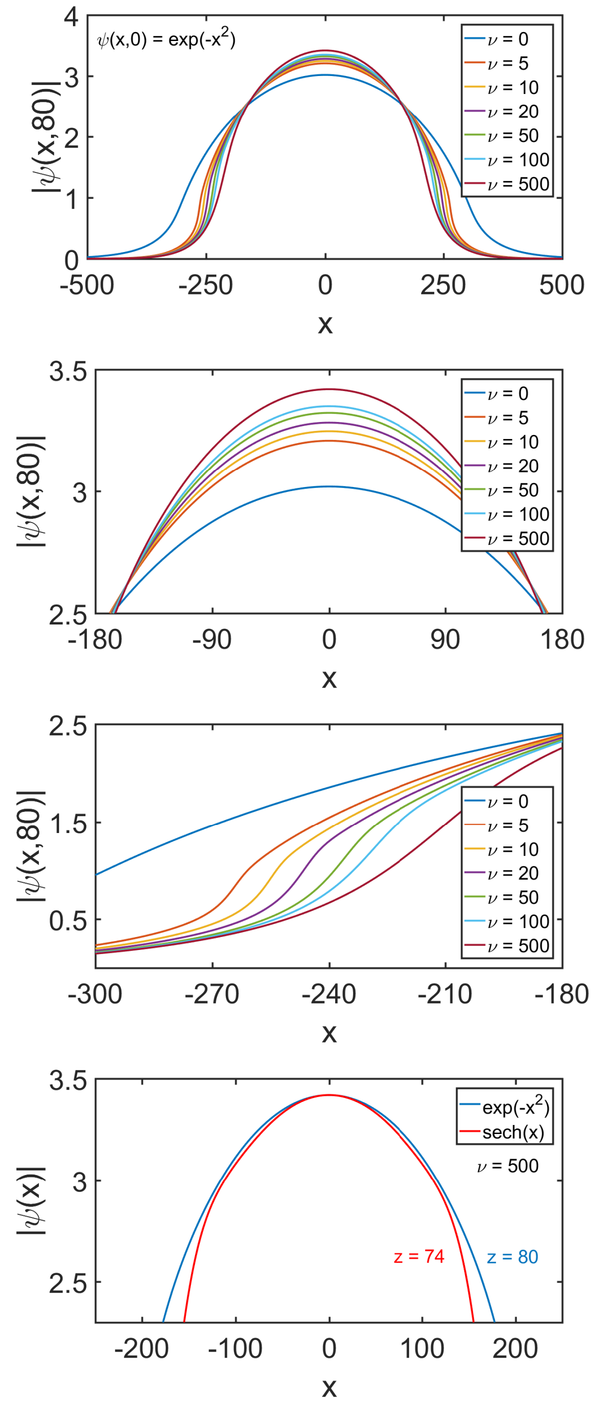


**Fig. 2.** (Color Online) Top three panels: The resulting beam for a Gaussian initial profile for different values of $\nu$ and $\gamma = 0.05$. Bottom panel: a zoom-in to illustrate the resulting parabolic profile, regardless of the initial shape and the relative distances needed to reach this state.

been used for other variations of the local NLS model [29–31]. In its current form the system, Eqs. (1), clearly dissipative, does not admit a Lagrangian density. However, it can be transformed [32] into a form for which a Lagrangian can be found by introducing the transformation $\psi(x,z) = u(x,z)\exp(\gamma z)$, so that the original

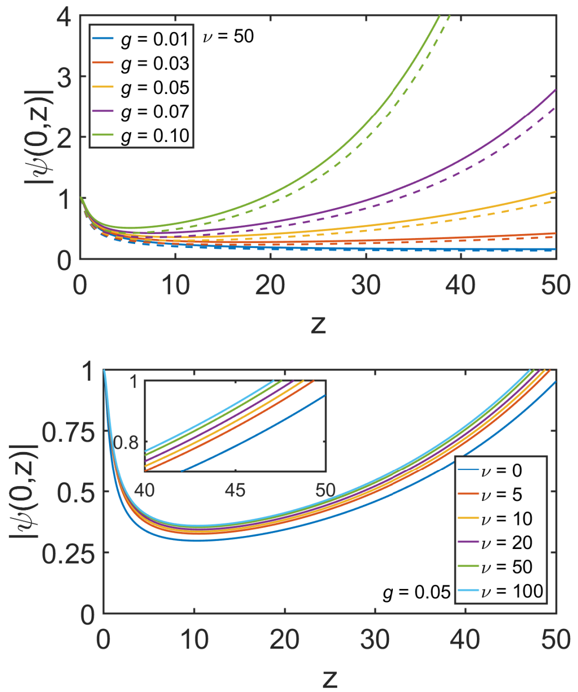


**Fig. 3.** (Color Online) Top: Evolution of the amplitude for different values of gain ($\nu = 50$); the dashed lines correspond to the relative NLS system ($\nu = 0$). Bottom: Evolution of the amplitude for different values of the nonlocality ($\gamma = 0.05$).

nonlocal system becomes

$$i\frac{\partial u}{\partial z} + \frac{d_0}{2}\frac{\partial^2 u}{\partial x^2} + 2\theta u = 0,$$
$$\nu\frac{\partial^2 \theta}{\partial x^2} - 2q\theta = -2e^{2\gamma z}|u|^2.$$

These equations have the Lagrangian density

$$\mathcal{L} = i(u^* u_z - u u_z^*) - d_0|u_x|^2 + 4\theta|u|^2 - \nu\theta_x^2 e^{-2\gamma z} - 2q\theta^2 e^{-2\gamma z}. \quad \textbf{(2)}$$

We now assume trial functions of the form [28, 29]

$$u(x,z) = A(z)\exp\{-b^2(z)[x - x_0(z)]^2 + i\phi(x,z)\}, \quad \textbf{(3)}$$
$$\theta(x,z) = \theta_0(z)\exp\{-b_2^2(z)[x - x_0(z)]^2\}, \quad \textbf{(4)}$$
$$\phi(x,z) = a_0(z) + a_1(z)x + a_2(z)x^2, \quad \textbf{(5)}$$

where we choose $b_2(z) = \sqrt{2}b(z)$ so that comparisons with the local NLS equation solution may be made, noting that if $\nu = 0$, $\theta = |\psi|^2/q$. We will comment further on this below. Next, we integrate (average) the density, Eq. (2), and find variations to obtain the system of ordinary differential equations (ODEs)

$$A' = -d_0 a_2 A, \quad \textbf{(6)}$$
$$b' = -2d_0 a_2 b, \quad \textbf{(7)}$$
$$x_0' = d_0(a_1 + 2a_2 x_0), \quad \textbf{(8)}$$
$$a_2' = 2d_0(b^4 - a_2^2) - \sqrt{2}\frac{b^2 e^{2\gamma z}}{q + \nu b^2}A^2, \quad \textbf{(9)}$$
$$a_1' = -2d_0\left(a_1 a_2 + 2b^4 x_0\right) + 2\sqrt{2}\frac{b^2 x_0 e^{2\gamma z}}{q + \nu b^2}A^2, \quad \textbf{(10)}$$
$$a_0' = -\frac{d_0}{2}\left(a_1^2 - 4b^4 x_0^2 + 2b^2\right) - \frac{(4b^2 x_0^2 - 5)\, e^{2\gamma z}}{2\sqrt{2}\,(q + \nu b^2)}A^2, \quad \textbf{(11)}$$

where $\theta_0(z) = A^2\exp(2\gamma z)/(q + \nu b^2)$ and the prime denotes differentiation with respect to $z$. Note here that a general expression for $b_2(z)$ (which can also be obtained algebraically as for $\theta_0(z)$) can also be used, but this does not change the results obtained in any meaningful way. The first two equations reveal the relationship between height $A$ and (inverse) width $b$ of the beam. Indeed, eliminating $a_2$, it is trivial to show that $b = cA^2$, where $c = b(0)/A^2(0)(= 1$ for our purposes here). Furthermore, we fix the center of the beam to be $x_0 = 0$, which also suggests that $a_1 = 0$. Thus, all parameters can be written in terms of the amplitude $A$ and, as such, we only require an equation for it. After some manipulation, it is found that the evolution of $A$ is governed by

$$A'' - \frac{3A'^2}{A} + 2d_0^2 c^4 A^9 - \frac{2c^2 d_0 e^{2\gamma z}}{\sqrt{2}(q + c^2\nu A^4)}A^7 = 0. \quad \textbf{(12)}$$

This is supplemented by the initial conditions $A(0) = 1$ and $A'(0) = 0$, so that Eq. (6) is also satisfied. While Eq. (12) may be difficult, or even impossible, to solve analytically, two interesting observations can be made. The first is that one can numerically show that all solutions are bounded by the curves corresponding to the extreme values of $\nu$, namely $\nu = 0$ (corresponding to the local NLS limit) and $\nu \to \infty$. This is demonstrated in Fig. 4, where we plot the amplitude corresponding to the original system, Eqs. (1), for various values of $\nu \in [0, +\infty)$. It is noted that over a broad range $\nu = O(10)$–$O(100)$ the amplitudes are nearly identical. As this range is the typical experimental range for milliwatt power beams in nematics [33], we see that the self-similar solution is almost a unique attractor for experimental regimes.

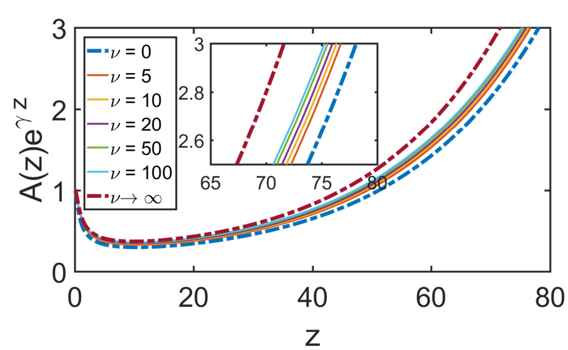


**Fig. 4.** (Color Online) The amplitude growth as given by the solutions of Eq. (12) for different values of $\nu$. The dashed-dotted thicker lines correspond to the two extreme values of the nonlocality, namely $\nu = 0$ (blue) and $\nu \to \infty$ (red).

Clearly, the weakest growth is exhibited in the local NLS limit, with it increasing as $\nu$ increases. In fact, in the limit $\nu \to \infty$, the amplitude Eq. (12) may be simplified to

$$A'' - \frac{3A'^2}{A} + 2d_0^2 c^4 A^9 = 0, \tag{13}$$

which possesses the exact solution

$$A(z) = A(0)\left(1 + 4c^4 d_0^2 z^2\right)^{-1/4}. \tag{14}$$

While this solution is valid in the limit $\nu \to \infty$, it provides an important qualitative feature of the growth rate, other than being exponential. The decay of the beam at its early stage of evolution follows an algebraic law of the form $\sim z^{-1/2}$.

We conclude our analysis with an important observation. The variational equations (6) and (7) also relate to the power of the beam $P = \int_{-\infty}^{\infty} |\psi|^2\, dx$. It is straightforward to show that its dependence on the propagation distance $z$ is given by $P' = 2\gamma P$, so that $P(z) = P(0)\exp(2\gamma z)$. As this is an exact result, it holds for any beam profile $\psi(x,z)$, and thus for our Gaussian. After a simple integration, we obtain

$$P = \sqrt{\frac{\pi}{2}}\frac{A^2}{b}e^{2\gamma z} = E(0)e^{2\gamma z}, \quad \frac{A^2}{b} = 1/c, \quad b = cA^2.$$

This shows that there is an inverse relationship between the amplitude and the width of the beam. It also provides the rate they vary and, more importantly, this is independent of the nonlocality parameter $\nu$. Thus, in the study of the evolution of the beam with gain, one needs only look at the amplitude of the beam as all the other parameters depend on this.

A final comment can be made about the shape of the similariton. By definition, such beams have a parabolic profile which is preserved during propagation. This profile is [34]

$$\psi(x,z) = a(z)\sqrt{1 - x^2/w^2(z)}\,H[w(z) - |x|], \tag{15}$$

where $H(x)$ is the Heaviside step function. This profile is also preserved here, as seen in Fig. 5. It is seen that in the nonlocal limit the parabolic profile is nearly independent of the nonlocality parameter $\nu$.

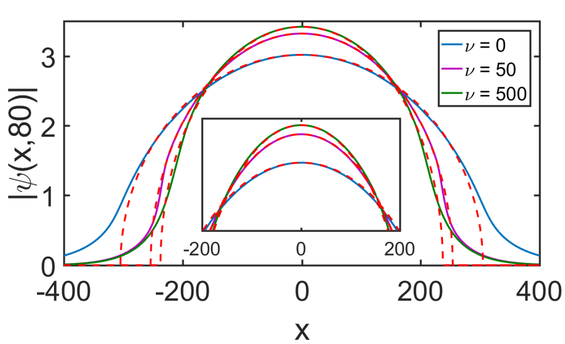


**Fig. 5.** (Color Online) The shape (and a zoom-in around the maxima) of the beam at $z = 80$ approximated by a parabolic profile of the form of Eq. (15).

To conclude, we have studied the propagation of similaritons in general nonlocal, nonlinear media. It is found that an input beam will evolve to a (nearly) fixed beam profile in the nonlocal limit. Furthermore, the amplitude of the beam is bounded by the solution in the two limiting cases, $\nu = 0$ (the local NLS limit) from below and the high nonlocality limit $\nu \to \infty$ from above. In the latter limit there is an exact solution of the variational equations for the beam. One of the major problems with the nematic equations (1) is the lack of any (known) exact solitary wave solutions. This work shows that in the presence of linear gain, a general beam will evolve to a fixed profile.

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